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# Performance of Finite Order Stochastic Process Generated Universal Portfolios

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## ABSTRACT

Stochastic processes based universal portfolio is a good generalisation universal portfolio which is believed be able to perform well with the right stochastic processes. The empirical performance of the stochastic process generated universal portfolio are analysed experimentally concerning 10 higher volume stocks from different categories in Kuala Lumpur Stock Exchange. The time interval of study is from January 2000 to December 2015, which includes the credit crisis from September 2008 to March 2009. A Constant Rebalanced Portfolio (CRP) is an investment strategy which reinvests by redistributing wealth equally among a set of stocks. The empirical performance of the finite-order universal portfolio generated by stochastic process shown to be better than Constant Rebalanced Portfolio with properly chosen parameters.

**Keywords:** Stochastic process, universal portfolio

# 1. Introduction

A finite-order universal portfolio generated by a set of independent Brownian motions is studied. Since a Brownian motion is also a Gaussian process, the joint distribution of the random variables at a set of distinct times is multivariate normal with the mean and covariance vectors depending on the drift coefficient, variance parameter and the sampled times. In the portfolio, the past price relatives are weighted by the joint moments of the Brownian motions which depend on the Brownian motion parameters and the sampled times.

For a weakly stationary process, a different type of universal portfolio is proposed where the weights on the stock prices depend only on the time differences of the stock prices. An empirical study is conducted on the returns achieved by the universal portfolios generated by the Ornstein-Uhlenbeck process on selected stock-price data sets.

The idea of using a probability distribution to generate a universal portfolio is due to Cover Cover (1991). The Cover-Ordentlich universal portfolioCover and Ordentlich (1996) is a moving-order universal portfolio. This moving-order universal portfolio are not practical in the sense that as the number of stocks in the portfolio increases, the implementation time and the computer storage requirements grow exponentially fast. Therefore, a finite-order universal portfolio generated by some probability distribution, due to Tan (2013) with comparable performance and requiring faster implementation time and much lesser computer memory is introduced. This type of universal portfolio depends only on the positive moments of the generating probability distribution. Tan and Pang (2013b) has studied the Multinomial generated universal portfolio and Tan and Pang (2013a) studied universal portfolio generated by Multivariate Normal Distribution.

We present an experimental study of two finite order stochastic process generated universal portfolios, namely the finite order Brownian-motion universal portfolio and the finite order universal portfolio generated by Ornstein-Uhlenbeck. Ten most active stocks data from Kuala Lumpur Stock Exchange with higher volume from different categories are selected from the top 100 listed companies. The day-end KLSE data was obtained from Yahoo (2016). The database contains daily opening prices, daily closing prices, daily high and low, and the volume of transaction. These ten stocks data with their respective code are shown in Table 1. The trading period is between January 2000 to December 2015. The above order one universal portfolios are run on every dataset consist of three stock data generated from the combination of these 10 selected most active stocks. The wealth achieved after the $n$ trading days by the above portfolio strategies is compared to the wealth obtained by CRP strategies. The well performing parameters of the above two universal portfolio strategies are observed.

Table 1: Ten most active stocks from different categories

| Category | stock code | stock name | Average Volume | Period |
|---|---|---|---|---|
| Construction | 5398 | Gamuda Bhd | 4986282.33 | January 2000 to December 2015 |
| Consumer Product | 7084 | QL Resources Berhad | 2086150 | March 2000 to December 2015 |
| Finance | 1818 | Bursa Malaysia Bhd | 51271350 | January 2005 to December 2015 |
| Hotel | 5517 | Shangri-La Hotels Malaysia Bhd | 69100 | January 2000 to December 2015 |
| Industry Products | 7106 | Supermax Corporation Bhd | 12656000 | January 2000 to December 2015 |
| IPC | 5031 | Time Dotcom Bhd | 630100 | March 2001 to January 2015 |
| Plantation | 2216 | IJM Plantation Bhd | 10168800 | July 2003 to December 2015 |
| Properties | 5148 | UEM Sunrise Bhd | 9561550 | January 2000 to December 2015 |
| Trading Services | 5099 | Air Asia Bhd | 94589600 | November 2004 to December 2015 |
| Trading Services | 3182 | Genting Bhd | 13869700 | January 2000 to December 2015 |

# 2. General Method For Universal Portfolio Generation

Consider an $m$-stock market. Let $\mathbf{x}_n = (x_{ni})$ be the stock-price-relative vector on the $n^{th}$ trading day, where $x_{ni}$ denotes the stock-price relative of stock $i$ on day $n$, which is defined to be the ratio of the closing price to its opening price on day $n$, for $i = 1, 2, \cdots, m$. Let $\hat{\mathbf{b}}_n = (\hat{b}_{n,i})$ denotes the universal portfolio vector on the $n^{th}$ trading day, where $\hat{b}_{ni}$ is the proportion of the current wealth on day $n$ invested on stock $i$, for $i = 1, 2, \cdots, m$ and $\sum_{i=1}^{m} b_{ni} = 1$. The initial wealth $\hat{S}_0$ is assumed to be one unit and the wealth at the end of $n^{th}$ trading day $\hat{S}_n$ is giving by

$$\hat{S}_n = \hat{\mathbf{b}}_1^t \mathbf{x}_1 \times \hat{\mathbf{b}}_2^t \mathbf{x}_2 \times \cdots \times \hat{\mathbf{b}}_n^t \mathbf{x}_n \tag{1}$$

where $\mathbf{b}^t$ denotes the transpose of the vector $\mathbf{b}$.

The theory of universal portfolio order $\nu$ generated by probability distribution is due to Tan (2013).

Let $Y_1, Y_2, \cdots, Y_m$ be $m$ discrete/continuous random variables having joint probability mass/density function $f(y_1, \cdots, y_m)$ defined over the domain $D$ where

$$D = \{(y_1, \cdots, y_m) : f(y_1, \cdots, y_m) > 0\} \tag{2}$$

Let $\{Y_{n1}\}_{n=1}^{\infty}$, $\{Y_{n2}\}_{n=1}^{\infty}$,$\cdots$, $\{Y_{nm}\}_{n=1}^{\infty}$ be $m$ given independent stochastic processes. For a fixed positive $\nu$, the $\nu$-order universal portfolio $\{\mathbf{b}_{n+1}\}$ generated by the $m$ given stochastic processes is defined as:

$$b_{n+1,k} = \frac{E[Y_{nk}(\mathbf{Y}_n^t \mathbf{x}_n)(\mathbf{Y}_{n-1}^t \mathbf{x}_{n-1}) \cdots (\mathbf{Y}_{n-(\nu-1)}^t \mathbf{x}_{n-(\nu-1)})]}{\sum_{j=1}^m E[Y_{nj}(\mathbf{Y}_n^t \mathbf{x}_n)(\mathbf{Y}_{n-1}^t \mathbf{x}_{n-1}) \cdots (\mathbf{Y}_{n-(\nu-1)}^t \mathbf{x}_{n-(\nu-1)})]} \tag{3}$$

for $k = 1, 2, \cdots, m$ and where the vector $\mathbf{Y}_l = (Y_{l1}, \cdots, Y_{lm})$ for $l = 1, 2, \cdots$

Expanding the numerator and denominator of (3), $b_{n+1,k}$ can be written as

$$\frac{\sum_{i_1=1}^m \sum_{i_2=1}^m \cdots \sum_{i_\nu=1}^m (x_{ni_1} x_{n-1,i_2} \cdots x_{n-\nu+1,i_\nu}) E[Y_{nk} Y_{ni_1} Y_{n-1,i_2} \cdots Y_{n-\nu+1,i_\nu}]}{\sum_{j=1}^m \{\sum_{i_1=1}^m \sum_{i_2=1}^m \cdots \sum_{i_\nu=1}^m (x_{ni_1} x_{n-1,i_2} \cdots x_{n-\nu+1,i_\nu}) E[Y_{nj} Y_{n_{i_1}} Y_{n-1,i_2} \cdots Y_{n-\nu+1,i_\nu}]\}} \tag{4}$$

## 2.1 Brownian-motion(Bm) Process Generated Universal Portfolio

From (4), $E[Y_{s_1 i_1} Y_{s_2 i_2} \cdots Y_{s_u i_u}] = E[Y_{s_1 i_1}] E[Y_{s_2 i_2}] \cdots E[Y_{s_u i_u}]$ if the $u$ integers $i_1, i_2, \cdots, i_u$ are distinct. Otherwise $E[Y_{s_1 j} Y_{s_2 j} \cdots Y_{s_u j}]$ is determined by using the moment-generating function of $Y_{s_i j}, Y_{s_2 j}, \cdots, Y_{s_u j}$.

Specifically,

$$E[(Y_{s_1 j} Y_{s_2 j} \cdots Y_{s_u j})(Y_{r_1 k} Y_{r_2 k} \cdots Y_{r_p k})] = E[Y_{s_1 j} Y_{s_2 j} \cdots Y_{s_u j}] \times E[Y_{r_1 k} Y_{r_2 k} \cdots Y_{r_p k}]$$

for $j \neq k$.

$$E\left[\prod_{i=1}^{q}(Y_{s_{i_1}j_i}Y_{s_{i_2}j_i}\cdots Y_{s_{i_{u_i}}j_i})\right]=\prod_{i=1}^{q}E(Y_{s_{i_1}j_i}Y_{s_{i_2}j_i}\cdots Y_{s_{i_{u_i}}j_i}) \tag{5}$$

for any set of distinct integers $j_1, j_2, \cdots, j_q$.

When $\{Y_{n1}\}_{n=1}^{\infty}$, $\{Y_{n2}\}_{n=1}^{\infty}$, $\cdots$, $\{Y_{nm}\}_{n=1}^{\infty}$ are independent Brownian motion with positive drift coefficients $\mu_1, \mu_2, \cdots, \mu_m$ and variance parameters $\sigma_1^2, \sigma_2^2, \cdots,$
$\sigma_m^2$ respectively.

According to Ross (2007), the process $\{Y_{nl}\}$ has stationary and independent increments, where $Y_{nl}$ has a normal distribution with mean $n\mu_l$ and variance $n\sigma_l^2$ for $l = 1, 2, \cdots, m$ and $n = 1, 2, \cdots$.

The covariance of $Y_{n_1 l}$ and $Y_{n_2 l}$ is given by

$$\text{Cov}(Y_{n_1 l}, Y_{n_2 l}) = n_1\sigma_l^2 \quad \text{for} \quad 0 < n_1 \leq n_2. \tag{6}$$

Furthermore, the $\nu$ random variables $Y_{n-\nu+1,l}, Y_{n-\nu+2,l}, \cdots, Y_{nl}$ have a joint multivariate normal distribution with mean vector $\boldsymbol{\mu}_l = (\mu_{l_1}, \mu_{l_2}, \cdots, \mu_{l_\nu}) > \mathbf{0}$ and $\boldsymbol{\nu} \times \boldsymbol{\nu}$ covariance matrix

$$K_l = \sigma_l^2 L = \sigma_l^2(\lambda_{ij}) \quad \text{for} \quad l = 1, 2, \cdots, m \tag{7}$$

where

$$\mu_{ik} = (n-\nu+k)\mu_i \quad \text{for} \quad k = 1, 2, \cdots, \nu \quad \text{and} \quad \lambda_{ij} = \begin{cases} n-\nu+i & \text{if } i \leq j, \\ n-\nu+j & \text{if } i > j. \end{cases}$$

for $i, j = 1, 2, \cdots, \nu$.

Note that the lambda matrix $L = (\lambda_{ij})$ in (7) does not depend on $l$. The components $\lambda_{ij}$ are the covariance components of $Y_{n-\nu+1,l}, \cdots, Y_{n,l}$ that depend on time $n$. The means $\mu_{ik}$ are nonnegative for $n \geq \nu = 1$, $k = 1, 2, \cdots, \nu$ and $l = 1, 2, \cdots, m$. Similarly, $\lambda_{ij} \geq 0$ for all $i, j = 1, 2, \cdots, \nu$ if $n \geq \nu - 1$.

In particular, when $\nu = 1$ and $m = 3$, (4) becomes

$$b_{n+1,k} = \frac{x_{n,1}E(Y_{nk}Y_{n1}) + x_{n,2}E(Y_{nk}Y_{n2}) + x_{n,3}E(Y_{nk}Y_{n3})}{\sum_{j=1}^{3}[x_{n,1}E(Y_{nj}Y_{n1}) + x_{n,2}E(Y_{nj}Y_{n2}) + x_{n,3}E(Y_{nj}Y_{n3})]} \tag{8}$$

for $k = 1, 2, 3$

Since the Brownian process are assumed to be independent, from (6) to (7), when $\nu = 1$, we have

$$\begin{aligned} &E(Y_{nk}^2) = n(\sigma_k^2 + n\mu_k^2) \\ &E(Y_{nk}Y_{ni}) = E(Y_{nk})E(Y_{ni}) = n^2\mu_k\mu_i \qquad \text{for} \quad k \neq i \\ &E(Y_{nk}Y_{nj}) = E(Y_{nk})E(Y_{nj}) = n^2\mu_k\mu_j \qquad \text{for} \quad k \neq j \end{aligned} \tag{9}$$

## 2.2 Ornstein Uhlenbeck(Oh) Process Generated Universal Portfolio

According to Ross (2007), a stochastic process $\{Y_r\}_{r=1}^{\infty}$ is said to be weakly stationary if $E(Y_r) = \mu$, independent of the time $r$ and $cov(Y_r, Y_{r+s})$ does not depend of the $r$ but depends on the time difference $s$ only. For $m$ given weakly stationary processes $\{Y_{n1}\}, \{Y_{n2}\}, \cdots \{Y_{nm}\}$, $E(\cdot)$ can be defined by rearranging the product of random variables $(Y_{nk}Y_{ni_1}Y_{n-1,i_2} \cdots Y_{n-\nu+1,i_\nu})$ as $m$ products, where in each product, the random variables come from the same process.

For weakly stationary process, $(Y_{nk}Y_{ni_1}Y_{n-1,i_2} \cdots Y_{n-\nu+1,i_\nu})$ can be written as the following $m$ products:

$$\begin{aligned} (Y_{nk}Y_{ni_1}Y_{n-1,i_2} \cdots Y_{n-\nu+1,i_\nu}) = (Y_{r_1 1}Y_{r_2 1} \cdots Y_{r_n 1})(Y_{u_1 2}Y_{u_2 2} \cdots Y_{u_n 2}) \\ \times (Y_{v_1 3}Y_{v_2 3} \cdots Y_{v_n 3}) \times \cdots \times (Y_{w_1 m}Y_{w_2 m} \cdots Y_{w_n m}) \end{aligned} \tag{10}$$

for some ordered sequences of time indices $r_1 \geq r_2 > \cdots > r_n; u_1 \geq u_2 > \cdots > u_n; v_1 \geq v_2 > \cdots > v_n; \cdots ; w_1 \geq w_2 > \cdots > w_n$. Taking expected value,

$$\begin{aligned} E(Y_{nk}Y_{ni_1}Y_{n-1,i_2} \cdots Y_{n-\nu+1,i_\nu}) = E(Y_{r_1 1}Y_{r_2 1} \cdots Y_{r_n 1})E(Y_{u_1 2}Y_{u_2 2} \cdots Y_{u_n 2}) \\ \times E(Y_{v_1 3}Y_{v_2 3} \cdots Y_{v_n 3}) \times \cdots \times E(Y_{w_1 m}Y_{w_2 m} \cdots Y_{w_n m}) \end{aligned} \tag{11}$$

where the functional $E$ is defined as :

$$E(Y_{q_1k}Y_{q_2k}\cdots,Y_{q_{n_q}k}) = \begin{cases} \prod_{i=1}^{n_q-1} E(Y_{q_ik}Y_{q_{i+1}k}) & \text{if } n_q \text{ is even,} \\ E(Y_{q_{n_q}k}) \prod_{i=1}^{n_q-1} E(Y_{q_ik}Y_{q_{i+1}k}) & \text{if } n_q \text{ is odd.} \end{cases} \tag{12}$$

for $k = 1, 2, \cdots, m; q_1 \geq q_2 > \cdots > q_{n_q}$.

The translated Ornstein-Uhlenbeck process $\{Y_r\}$ is define by $Y_r = Z_r + \mu$ for all $r$ is said to have parameters $(\mu, \sigma)$ if $E[Y_r] = \mu$ and $E[Y_rY_{r+s}] = e^{\frac{-\alpha s}{2}} + \mu^2$ for $s > 0$, $\alpha > 0$. It is also assumed that $\mu > 0$. Let $\{Y_{n1}\}, \{Y_{n2}\}, \cdots \{Y_{nm}\}$ be $m$ given independent (translated) Ornstein-Uhlenbeck process with parameters $(\mu_1, \alpha_1), (\mu_2, \alpha_2), \cdots, (\mu_m, \alpha_m)$ respectively., where all parameters are positive. Consider the universal portfolio (4) generated by these process where $E(\cdot)$ is defined by (11) and (12), namely

$$E(Y_{q_1k}Y_{q_2k}\cdots,Y_{q_{n_q}k}) = \begin{cases} \prod_{i=1}^{n_q-1} [e^{\frac{-\alpha_k(q_i-q_{i+1})}{2}} + \mu_k^2] & \text{if } n_q \text{is even,} \\ \mu_k \prod_{i=1}^{n_q-2} [e^{\frac{-\alpha_k(q_i-q_{i+1})}{2}} + \mu_k^2] & \text{if } n_q \text{ is odd.} \end{cases} \tag{13}$$

for $k = 1, 2, \cdots, m : q_1 \geq q_2 > \cdots > q_{n_q}$,

in particular, when $\nu = 1$, $m = 3$, (3) becomes

$$b_{n+1,k} = \frac{\sum_{i=1}^{3} x_{n,i}E(Y_{nk}Y_{n,i})}{\sum_{j=1}^{3}(\sum_{i=1}^{3} x_{n,i}E(Y_{nj}Y_{ni}))} \tag{14}$$

and the (13) becomes

$$\begin{aligned} &E(Y_{nk}Y_{nk}) = E(Y_{nk}^2) = 1 + \mu_k^2 \\ &E(Y_{nk}Y_{ni}) = E(Y_{nk})E(Y_{ni}) = \mu_k\mu_i \qquad \text{for} \quad i \neq k \\ &E(Y_{nk}Y_{nj}) = E(Y_{nk})E(Y_{nj}) = \mu_k\mu_j \qquad \text{for} \quad j \neq k \end{aligned} \tag{15}$$

for $k = 1, 2, 3$.

## 2.3 Constant Rebalanced Portfolio

A *constant-rebalanced portfolio* is a portfolio $\mathbf{b} = (b_i)$ that is constant over the trading days and the wealth at the end of $n$ trading days is

$$S(\mathbf{x}^n) = \prod_{i=1}^{n} \mathbf{b}^t \mathbf{x}_n. \quad (16)$$

The above two finite-order universal portfolios are studied on 10 most active stock-price data selected from the Kuala Lumpur Stock Exchange, which described in Introduction section. Every three stock data are generated from the 10 stocks data by using combination. In Tan and Pang (2014a) and Tan and Pang (2014b), good performances are obtained by order one Brownian-motion generated universal portfolio and order one universal portfolio generated by Ornstein Uhlenbeck process, with the wealth achieved outperform the Diriclet universal portfolio. Therefore, only order one of the proposed two universal portfolio strategies are used for the data analysis.

## 2.4 Parameters for Brownian-motion Generated Universal Portfolio

In Tan and Pang (2014a), good performances parameter of order one universal portfolio generated by Brownian-motion is (1,1.4;10,2.5; 100,3.6), therefore, the 10 sets of parameters are formed among (1,1.4; 10,2.5; 100,3.6), i.e. (1,1.4; 10,2.5; 100,3.6), (100,1.4;1,2.5;10,3.6), (100,1.4;10,2.5; 1,3.6), (10,1.4; 1,2.5; 100,3.6), (1,1.4;100,2.5; 10,3.6), (10,1.4;100,2.5; 1,3.6), (1,1.4;10,3.6; 100,2.5), (1,2.5; 10,1.4; 100,3.6), (1,2.5; 10,3.6; 100,1.4) and (1,3.6;10,1.4;100, 2.5).

## 2.5 Parameters for Universal Portfolio Generated by Ornstein Uhlenbeck Process

In Tan and Pang (2014b), one of the parameters selected of is (16, 0.1, 0.5, 0.2, 1.2, 2) due to good performance is obtained. The other 9 set of parameters we formed by varying among (16, 0.1, 0.5, 0.2, 1.2, 2). The 10 set of parameters are (16, 0.1, 0.5, 0.2, 1.2, 2), (16, 0.5, 0.1, 0.2, 1.2, 2), (0.5, 0.1, 16, 0.2, 1.2, 2), (0.5, 16, 0.1, 0.2, 1.2, 2), (0.1, 16, 0.5, 0.2, 1.2, 2), (0.1, 0.5, 16, 0.2, 1.2, 2), (16, 0.1, 0.5, 2, 1.2, 0.2), (16, 0.5, 0.1, 2, 1.2, 0.2), (50, 0.5, 2, 0.2, 1.2, 2) and (50, 2, 0.5, 0.2, 1.2, 2).

# 3. Result and Conclusion

The CRP wealth is a benchmark measurement for good performance. A comparison can be made with the wealth obtained by CRP and the wealth achieved by the universal portfolios generated by Brownian motion and Ornstein Uhlenbeck Process with the selected parametric vector stated in previous sections. Every one year interval starting from year 2000 to year 2015 of the available stock data listed in Table 1 are used for study. At least 20 percent of the wealth achieved by proposed universal portfolio performed better than CRP are obtained for analysis. The good performance of the parameter is observed.

The average wealth and standard deviation obtained by Brownian Motion generated universal portfolio are shown in Tables 2, 4 and 3. From the ten set of parameters chosen, the parameter performed well is (100,1.4; 1,2.5; 10,3.6) which has a higher frequency in above tables. From Tables 2, 4 and 3, parameter (100,1.4; 1,2.5; 10,3.6) has achieved good average wealth when compared to CRP with the range from 2.0265 to 604.4073.

Next, the average wealth and standard deviation obtained by the Ornstein Uhlenbeck Process generated universal portfolio are shown in Tables 5, 6 and 7. From these tables, among the ten set of parameters chosen, the parameter performed well for universal portfolio generated by Ornstein Uhlenbeck process is (16,0.1,0.5,0.2,1.2,2) with the higher frequency observed. Also, the average wealth obtained by this set of parameter when compared to CRP is from range 2.1214 to 762.8233. Hence, for future research, the empirical study of performances of Malaysia stocks will be studied by using the above two universal portfolios with their well performed parameters identified in the above results obtained.

Table 2: Average wealth obtained by Brownian Motion Generated Universal Portfolio better than CRP

| Parameter for Brownian motion strategy | Duration | Average Wealth | Standard Deviation |
|---|---|---|---|
| (100,1.4;1,2.5;10,3.6) | 1 Jan 2015 ∼ 31 Dec 2015 | 2.0265425 | 0.079012819 |
| (100,1,4;10,2.5;1,3.6) | 1Jan 2015 ∼ 31 Dec 2015 | 2.023856 | 0.070359953 |
| (100,1.4;1,2.5;10,3.6) | 1 Jan 2014 ∼ 31 Dec 2015 | 7.3190495 | 1.146490914 |
| (100,1.4;10,2.5;1,3.6) | 1 Jan 2014 ∼ 31 Dec 2015 | 7.30123 | 1.118149367 |
| (100,1.4;1,2.5;10,3.6) | 1 Jan 2013 ∼ 31 Dec 2015 | 30.064781 | 8.165413136 |
| (100,1.4;10,2.5;1,3.6) | 1 Jan 2013 ∼ 31 Dec 2015 | 29.811969 | 7.895890902 |
| (100,1.4;1,2.5;10,3.6) | 1 Jan 2008 ∼ 31 Dec 2015 | 235.709762 | 70.87220535 |
| (100,1.4;10,2.5;1,3.6) | 1 Jan 2008 ∼ 31 Dec 2015 | 238.609657 | 75.33569953 |

Table 3: Average wealth obtained by Bm Generated Universal Portfolio better than CRP

| Strategies | Duration | Average Wealth | Standard Deviation |
|---|---|---|---|
| Brownian-motion(1,1.4;100,2.5;10,3.6) | 1 Jan 2001 ∼ 31 Dec 2015 | 22.2905245 | 0.497273551 |
| Brownian-motiom (1,1.4;10,2.5;100,3.6) | 1 Jan 2000 ∼ 31 Dec 2015 | 19.3637865 | 6.39364608 |
| Brownian-motion (1,1.4;10,3.6;100,2.5) | 1 Jan 2000 ∼ 31 Dec 2015 | 18.075308 | 4.570732577 |
| Brownian-motion (1,2.5;10,1.4;100,3.6) | 1 Jan 2000 ∼ 31 Dec 2015 | 19.3638495 | 6.393660223 |
| Brownian-motion (1,2.5;10,3.6;100,1.4) | 1 Jan 2000 ∼ 31 Dec 2015 | 19.364966 | 6.394042767 |
| Brownian-motion (1,3.6;10,1.4;100,2.5) | 1 Jan 2000 ∼ 31 Dec 2015 | 19.36463 | 6.393915488 |
| Brownian-motion (1,3.6;10,2.5;100,1.4) | 1Jan 2000 ∼ 31 Dec 2015 | 19.365065 | 6.394063981 |
| Brownian-motion (10,1.4;1,2.5;100,3.6) | 1 Jan 2000 ∼ 31 Dec 2015 | 18.3479715 | 4.638068234 |

Table 4: Average wealth obtained by Bm Generated Universal Portfolio better than CRP

| Strategies | Duration | Average Wealth | Standard Deviation |
|---|---|---|---|
| Brownian-motion (100,1.4;1,2.5;10,3.6) | 1 Jan 2007 ∼ 31 Dec 2015 | 205.250947 | 60.58070031 |
| Brownian-motion (100,1.4;10,2.5;1,3.6) | 1 Jan 2007 ∼ 31 Dec 2015 | 201.434666 | 55.46979897 |
| Brownian-motion (100,1.4;1,2.5;10,3.6) | 1 Jan 2006 ∼ 31 Dec 2015 | 278.414544 | 89.7088738 |
| Brownian-motion (100,1.4;10,2.5;1,3.6) | 1 Jan 2006 ∼ 31 Dec 2015 | 271.118674 | 82.66402834 |
| Brownian-motion (100,1.4;1,2.5;10,3.6) | 1 Jan 2005 ∼ 31 Dec 2015 | 384.4106285 | 137.6970185 |
| Brownian-motion (100,1.4;10,2.5;1,3.6) | 1 Jan 2005 ∼ 31 Dec 2015 | 382.468823 | 137.9797162 |
| Brownian-motion (100,1.4;1,2.5;10,3.6) | 1 Jan 2004 ∼ 31 Dec 2015 | 604.4072875 | 154.8509637 |
| Brownian-motion (100,1.4;10,2.5;1,3.6) | 1 Jan 2004 ∼ 31 Dec 2015 | 595.479407 | 148.4204222 |
| Brownian-motion (1,1.4;10,2.5;100,3.6) | 1 Jan 2003 ∼ 31 Dec 2015 | 50.535135 | 13.37465041 |
| Brownian-motion (1,1.4;10,3.6;100, 2.5) | 1 Jan 2003 ∼ 31 Dec 2015 | 50.534848 | 13.37537307 |
| Brownian-motion (1,2.5;10,1.4;100,3.6) | 1 Jan 2003 ∼ 31 Dec 2015 | 50.53511 | 13.37392067 |
| Brownian-motion (1,2.5;10,3.6;100,1.4) | 1 Jan 2003 ∼ 31 Dec 2015 | 50.5346405 | 13.37510508 |
| Brownian-motion (1,3.6;10,1.4;100,2.5) | 1 Jan 2003 ∼ 31 Dec 2015 | 50.5347845 | 13.37350136 |
| Brownian-motion (1,3.6;10,2.5;100,1.4) | 1 Jan 2003 ∼ 31 Dec 2015 | 50.5346015 | 13.37396381 |
| Brownian-motion (10,1.4;1,2.5;100,3.6) | 1 Jan 2003 ∼ 31 Dec 2015 | 45.0898545 | 5.490426062 |

Table 5: Average wealth obtained by Universal Portfolio generated by Oh process better than CRP

| Strategies | Duration | Average Wealth | Standard Deviation |
|---|---|---|---|
| Ornstein [16,0.1,0.5,0.2,1.2,2] | 1 Jan 2015 ∼ 31 Dec 2015 | 2.121387 | 0.04412912 |
| Ornstein [16,0.5,0.1,0.2,1.2,2] | 1Jan 2015 ∼ 31 Dec 2015 | 2.1186715 | 0.035085931 |
| Ornstein [50,0.5,2,0.2,1.2,2] | 1 Jan 2015 ∼ 31 Dec 2015 | 2.111905 | 0.047320293 |
| Ornstein [50,2,0.5,0.2,1.2,2] | 1 Jan 2015 ∼ 31 Dec 2015 | 2.109227 | 0.038255891 |
| Ornstein [16,0.1,0.5,0.2,1.2,2] | 1 Jan 2014 ∼ 31 Dec 2015 | 8.067012 | 0.570505065 |
| Ornstein [16,0.5,0.1,0.2,1.2,2] | 1Jan 2014 ∼ 31 Dec 2015 | 8.04818 | 0.536612023 |
| Ornstein [50,0.5,2,0.2,1.2,2] | 1 Jan 2014 ∼ 31 Dec 2015 | 7.9887175 | 0.630782382 |
| Ornstein [50,2,0.5,0.2,1.2,2] | 1 Jan 2014 ∼ 31 Dec 2015 | 7.9701295 | 0.594828831 |
| Ornstein [16,0.1,0.5,0.2,1.2,2] | 1 Jan 2013 ∼ 31 Dec 2015 | 34.254638 | 3.75991303 |
| Ornstein [16,0.5,0.1,0.2,1.2,2] | 1Jan 2013 ∼ 31 Dec 2015 | 33.9694685 | 3.468970409 |
| Ornstein [50,0.5,2,0.2,1.2,2] | 1 Jan 2013 ∼ 31 Dec 2015 | 33.795792 | 4.234245915 |
| Ornstein [50,2,0.5,0.2,1.2,2] | 1 Jan 2013 ∼ 31 Dec 2015 | 33.5046435 | 3.927663507 |

Table 6: Average wealth obtained by Universal Portfolio generated by Oh process better than CRP

| Strategies | Duration | Average Wealth | Standard Deviation |
|---|---|---|---|
| Ornstein [50,0.5, 2,0.2,1.2,2] | 1 Jan 2009 ∼ 31 Dec 2015 | 189.172132 | 101.8969 |
| Ornstein [50,2,0.5,0.2,1.2,2] | 1 Jan 2009 ∼ 31 Dec 2015 | 200.759345 | 117.3114418 |
| Ornstein [16,0.1,0.5,0.2,1.2,2] | 1 Jan 2008 ∼ 31 Dec 2015 | 301.439745 | 35.3349362 |
| Ornstein [16,0.5,0.1,0.2,1.2,2] | 1 Jan 2008 ∼ 31 Dec 2015 | 305.113613 | 41.01533286 |
| Ornstein [50,0.5,2,0.2,1.2,2] | 1 Jan 2008 ∼ 31 Dec 2015 | 293.96517 | 39.45733479 |
| Ornstein [50,2,0.5,0.2,1.2,2] | 1 Jan 2008 ∼ 31 Dec 2015 | 297.5198085 | 45.04923068 |
| Ornstein [16,0.1,0.5,0.2,1.2,2] | 1 Jan 2007 ∼ 31 Dec 2015 | 266.2532905 | 35.79009871 |
| Ornstein [16,0.5,0.1,0.2,1.2,2] | 1 Jan 2007 ∼ 31 Dec 2015 | 264.900757 | 34.06162922 |
| Ornstein [50,0.5, 2,0.2,1.2,2] | 1 Jan 2007 ∼ 31 Dec 2015 | 259.4439375 | 38.96697109 |
| Ornstein [50,2,0.5,0.2,1.2,2] | 1 Jan 2007 ∼ 31 Dec 2015 | 257.844673 | 36.90749077 |
| Ornstein [16,0.1,0.5,0.2,1.2,2] | 1 Jan 2006 ∼ 31 Dec 2015 | 363.7716645 | 50.53144935 |
| Ornstein [16,0.5,0.1,0.2,1.2,2] | 1 Jan 2006 ∼ 31 Dec 2015 | 358.3183775 | 47.82847994 |
| Ornstein [50,0.5,2,0.2,1.2,2] | 1 Jan 2006 ∼ 31 Dec 2015 | 353.992774 | 54.97640672 |
| Ornstein [50,2,0.5,0.2,1.2,2] | 1 Jan 2006 ∼ 31 Dec 2015 | 347.856213 | 51.40405377 |

Table 7: Average wealth obtained by Universal Portfolio generated by Oh process better than CRP

| Strategies | Duration | Average Wealth | Standard Deviation |
|---|---|---|---|
| Ornstein [16,0.1,0.5,0.2,1.2,2] | 1 Jan 2005 ∼ 31 Dec 2015 | 511.4671185 | 78.54948217 |
| Ornstein [16,0.5,0.1,0.2,1.2,2] | 1 Jan 2005 ∼ 31 Dec 2015 | 509.6434985 | 80.83019004 |
| Ornstein [50,0.5,2,0.2,1.2,2] | 1 Jan 2005 ∼ 31 Dec 2015 | 497.090707 | 85.22536709 |
| Ornstein [50,2,0.5,0.2,1.2,2] | 1 Jan 2005 ∼ 31 Dec 2015 | 494.9842985 | 87.83733822 |
| Ornstein [16,0.1,0.5,0.2,1.2,2] | 1 Jan 2004 ∼ 31 Dec 2015 | 771.5952125 | 80.67172882 |
| Ornstein [16,0.5,0.1,0.2,1.2,2] | 1 Jan 2004 ∼ 31 Dec 2015 | 762.823391 | 71.85391563 |
| Ornstein [50,0.5,2,0.2,1.2,2] | 1 Jan 2004 ∼ 31 Dec 2015 | 752.528413 | 88.88544513 |
| Ornstein [50,2,0.5,0.2,1.2,2] | 1 Jan 2004 ∼ 31 Dec 2015 | 742.4438335 | 78.66768072 |
| Ornstein [0.1,0.5,16,0.2,1.2,2] | 1 Jan 2003 ∼ 31 Dec 2015 | 57.5212825 | 5.81661442 |
| Ornstein [0.5,0.1,16,0.2,1.2,2] | 1 Jan 2003 ∼ 31 Dec 2015 | 55.8499615 | 3.217462427 |
| Ornstein [0.5,16,0.1,0.2,1.2,2] | 1 Jan 2003 ∼ 31 Dec 2015 | 56.7503485 | 6.569207966 |
| Ornstein [0.1,0.5,16,0.2,1.2,2] | 1 Jan 2002 ∼ 31 Dec 2015 | 27.1964845 | 1.721150938 |
| Ornstein [0.5,0.1,16,0.2,1.2,2] | 1 Jan 2002 ∼ 31 Dec 2015 | 26.3251285 | 0.779508152 |
| Ornstein [0.5,16,0.1,0.2,1.2,2] | 1 Jan 2002 ∼ 31 Dec 2015 | 26.976849 | 1.914191797 |

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